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REGULAR p-GROUPS AND WORDS GIVING RISE TO COMMUTATIVE GROUP OPERATIONS

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ABSTRACT

Groups with words which give rise to commutative operations are considered. Finite p-groups with this property are shown to be regular. It is also shown that finite p-groups of class less than p, among other groups, have this property.

If G is a group, we may define a binary operation $*$ on G by

$$x * y = v(x, y)$$

where v is some fixed word in x and y. If the elements of G, together with the operation $*$, define a group, then we say that v is a group-word for G and write the corresponding group G_v.

A number of authors have considered the existence of group-words for groups, or classes of groups, and the relationship between G and G_v (see, for example, Higman and Neumann [2], Hulanicki and Świerczkowski [3] and Street [6]). In this note, we shall be interested in the problem, raised by Cooper [1], of when a group G possesses a group-word v for which G_v is abelian; following Cooper, we say that G is then verbally abelian. Much of the interest of verbally abelian groups lies in their connection with regular p-groups.

THEOREM 1. *Let G be a finite p-group. There is a group-word v for which G_v is regular if and only if G is regular.*

In particular, G is verbally abelian only if G and all its direct powers are regular (that is, G is V-regular in the sense of Weichsel [7]). We observe that V-regularity, rather than regularity, is appropriate here because direct powers of verbally abelian groups are again verbally abelian. This answers a question of Cooper in [1].

It remains to decide which V-regular groups are verbally abelian. We prove that at least those of "small" class are.

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THEOREM 2. *Suppose that G is a nilpotent group of class c and that the derived group of G has finite exponent n, where every prime divisor of n is greater than c. Then G is verbally abelian.*

(Note that this contradicts an example of a non-verbally-abelian group of small class claimed by Cooper in [1].)

The theorem is deduced from the following, apparently more general, lemma. If G is a group, let $\Lambda_n(G)$ denote the subgroup generated by all elements of order dividing n and $M_n(G)$ the subgroup generated by all nth powers of elements of G; let G' denote the derived group of G.

LEMMA. *Let F be a nilpotent group with a law*

$$x_1^n x_2^n = (x_1 x_2 w(x_1, x_2))^n$$

where w is a commutator word. Then $F/\Lambda_n(F')M_n(F')$ is verbally abelian.

We deduce Theorem 2 from this lemma by observing that the free nilpotent group of class c has such a law if every prime divisor of n is greater than c and that groups of the type described in Theorem 2 can be expressed as suitable quotients of these free nilpotent groups.

It is of interest to speculate about V-regular p-groups which are not of small class. Are they expressible as suitable quotients as required by the lemma? Are they even expressible as quotients of torsion-free groups which have a law of the required kind? If all V-regular groups are not verbally abelian, where is the dividing line? Theorem 2 suggests that this dividing line and the dividing lines for verbally-class-k groups or verbally-small-class groups may hopefully give partitions of the V-regular groups which are of some interest.

For basic results concerning regular groups, we refer to Huppert [4, III]. A section of a group is a homomorphic image of a subgroup.

PROOF OF THEOREM 1. If G is regular, there is nothing to prove. Suppose, then, that G is irregular and that G has a group-word v such that G_v is regular. Then G has a minimal irregular section H and clearly v is still a group-word for H and H_v is still regular.

It is straightforward to show that v is equivalent in H to a word

$$x_1 x_2 w(x_1, x_2)$$

where $w(x_1, x_2)$ is a commutator word in x_1 and x_2. Now, since H_v is regular, for all $x, y \in H_v$

$$(x * y)^{*p} = (x^{*p}) * (y^{*p}) * d_*^{*p}$$

where $*$ is the derived operation and x^{*p} denotes the pth power of x under this operation and d_* is a product of $*$-commutator words in x and y.

Now $x^{*p} = x^p$ and every $*$-commutator lies in H' (by a straightforward calculation). Hence, for x, $y \in H$,

$$(x * y)^p = x^p * y^p * d_*^p.$$

But, M is a minimal irregular group and so, by theorem 2(b) of Mann [5], $d_*^p = 1$. Hence

$$(xyw(x, y))^p = x^p y^p w(x^p, y^p).$$

But, by theorem $2(k)$ of Mann [5], $(xyw(x, y))^p = (xy)^p$ and $w(x^p, y^p) = 1$. Hence, for all x, $y \in H$,

$$(xy)^p = x^p y^p$$

and H is regular. This contradiction completes the proof of the theorem.

Proof of the Lemma. Although the lemma and Theorem 2 are based on considerations of finite p-groups, we have stated them in a more general form. We therefore list some consequences of the law $x_1^n x_2^n = (x_1 x_2 d(x_1, x_2))^n$ which can be proved in a manner similar to the analogous results for regular p-groups (see Huppert [5, III]):

1) $\Lambda_n(F)$ consists of elements of order dividing n;
2) $M_n(F)$ consists of nth powers;
3) if a, $b \in F$ and $a^n = b^n$, then $(ab^{-1})^n = 1$;
4) if a, $b \in F$, then there exists $c \in F'$ such that $[a^n, b^n] = c^{n^2}$.

Let x, y, z be elements of F. Define an operation $*$ on F by

$$a * b = abw(a, b)$$

where a, $b \in F$ and w is the commutator word given in the statement of the lemma. Then

$$x^n y^n z^n = \begin{cases} (x^n y^n) z^n = (x * y)^n z^n = ((x * y) * z)^n \\ x^n (y^n z^n) = x^n (y * z)^n = (x * (y * z))^n. \end{cases}$$

Hence $((x * y) * z)^n = (x * (y * z))^n$ and so, by (3) above,

$$(x * y) * z = x * (y * z) \cdot a$$

where $a \in \Lambda_n(F')$. Thus $*$ induces an associative operation on $F/\Lambda_n(F')$ and so on $F/\Lambda_n(F')M_n(F')$. That this operation in fact turns $F/\Lambda_n(F')M_n(F')$ into a group is now straightforward to check.

Now,

$$(x * y)^n = x^n y^n, \qquad (y * x)^n = y^n x^n$$

and so

$$(y * x)^{-n} (x * y)^n = [x^n, y^n] = d^{n^2}$$

for some $d \in F'$ (by (4) above). Hence

$$(y * x)^{-n} (x * y)^n \in \mathrm{M}_{n^2}(F')$$

and so, by (a) and (3) above,

$$((y * x)^{-1} (x * y))^n = e^{n^2} \qquad (e \in F').$$

Hence, by (3),

$$(e^{-n} (y * x)^{-1} (x * y))^n = 1.$$

Thus

$$(y * x)^{-1} (x * y) \in \Lambda_n(F')\mathrm{M}_n(F')$$

and so $*$ induces a commutative group operation on $F/\Lambda_n(F')\mathrm{M}_n(F')$. Hence $F/\Lambda_n(F')\mathrm{M}_n(F')$ is verbally abelian, as required.

Proof of Theorem 2. Let N be the free nilpotent group of class c and of rank the cardinality of a generating set of G. Since every prime divisor of n is greater than c, it follows from the Hall commutator collecting process (Huppert [5, III, 9]) that N has a law

$$(x_1 x_2)^n = x_1^n x_2^n d_1^n \cdots d_k^n$$

where $d_1, \cdots, d_k$ are commutator words. It is straightforward to deduce that N has a law

$$x_1^n x_2^n = (x_1 x_2 w(x_1, x_2))^n$$

where w is a commutator word.

Thus the lemma applies to N. Since N is torsion-free, $\Lambda_n(N') = \{1\}$ and so N/M_n is verbally abelian. But, since G has class c and G' has exponent n, G is a homomorphic image of $N/\mathrm{M}_n(N')$. Thus G is verbally abelian.

References

1. C. D. H. Cooper, *Words which give rise to another group operation for a given group*, in *Proc Internat. Conf. Theory of Groups* (Canberra 1965), 221–225. (*Lecture Notes in Mathematics*, No. 372 Springer-Verlag, Berlin, Heidelberg and New York, 1974.)

2. G. Higman and B. H. Neumann, *Groups as groupoids with one law*, Publ. Math. Debrecen **2** (1951–2), 215–221.

3. A. Hulanicki and S. Świerczkowski, *On group operations other than xy or yx*, Publ. Math. Debrecen **9** (1962), 142–148.

4. B. Huppert, *Endliche Gruppen* I, in *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Band **37**, Springer-Verlag, Berlin, Heidelberg and New York, 1967.

5. Avinoam Mann, *Regular p-groups*, Israel J. Math. **10** (1971), 471–477.

6. Anne Penfold Street, *Subgroup determining functions on groups*, Illinois J. Math. **12** (1968), 99–120.

7. Paul M. Weichsel, *Regular p-groups and varieties*, Math. Z. **95** (1967), 223–231.

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